Symplectic Effect for the Numerical Solution of Conservative Systems

Rafay Mustafa
HITEC University, Cantt, Taxila, Rawalpindi, Punjab 47080, Pakistan.
Email: rafay.mustafa@hitecuni.edu.pk

Received: 06-06-2022; **Accepted**: 23-08-2022; **Published**: 29-08-2022

Abstract: The focus of this paper is to use those numerical tools for conservative systems that provides an approximation flow for the Hamiltonian system, which defines a worldwide physical systems including planetary motion, simple Pendulum and several models. It has been observed that the symplectic scheme is found very effective for astronomical many body systems. We are particularly interested in those numerical schemes that possess the qualitative behavior of such systems and symplecticity of the flow. In this paper, we investigate the Hamiltonian systems for its symplecticity and G-symplecticity numerically and show explicitly how these techniques be effective for the preservation of energy. Since we have applied this scheme for the planetary body motion and found that the results are very much effective and the energy preserves during the motion of the planetary bodies. Since the aim of this paper is to investigate the energy preservation adopted by the symplectic methods.

Keywords: Hamiltonian, Symplecticity, G-symplecticity, Energy preservation

1 Introduction

Differential equations have a remarkable property to describe the world around us. It is a powerful tool for analyzing the relationship between several dynamic quantities. Consider the explicit differential equation,

$$\frac{dx}{dt} = F\big(t, x(t)\big)$$

Generally, if the equation is not depending on time variable t its autonomous form is

$$\frac{dx}{dt} = F\big(x(t)\big)$$

Usually, the solution of a differential equation depends upon its existence and uniqueness property defined as Lipschitz condition [1].

The equations defined as

$$\frac{dp}{dt} = -\frac{\partial H}{\partial r} \quad , \qquad \frac{dr}{dt} = \frac{\partial H}{\partial p}$$

are called the Hamiltonian equations. The concept of Hamiltonian H is introduced in 1824 by Hamilton [3]. These equations normally deal the equations of motion in terms of coordinates system. Hamiltonian systems (i.e., equations of motion) has the remarkable conservative property as

$$\frac{dH}{dt} = 0$$

The system of Hamiltonian states that the path of a flow remains constant [5,9]. This means that the oriented area founded by the vectors (p,r) of the Hamiltonian system is preserved [10]. Let ψ_H is the operator of the Hamiltonian equations given as,

$$\Psi\colon \big(p(0), r(0)\big) \mapsto \big(p(t), r(t)\big)$$

Copyright © 2022 IJAIMS

Where,

$$f = \begin{bmatrix} -\dfrac{\partial H}{\partial r} \\ \dfrac{\partial H}{\partial p} \end{bmatrix}$$

which is true for the Hamiltonian system i.e. $div\, f = 0$.

Therefore, ψ is symplectic.

2 Runge--Kutta Method

Runge–Kutta methods follows one-dimension methodology given by the Runge and Kutta in 1900. For the numerical solution of differential equation, this methodology will provide such approximations y_{n-1} that leads y_n in a single path. The R-K methods classified as implicit as well as explicit methods for the solutions of ODEs [1,2,3]. A standard s-stage explicit form of R-K method is defined as

$$Y_i = y_n + h\sum_{j=1}^{s} a_{ij} f\left(x_n + c_i h, Y_j\right), \quad i = 1, 2, \cdots, s,$$

$$Y_{n+1} = y_n + h\sum_{j=1}^{s} b_j f\left(x_n + c_j h, Y_j\right)$$

here a represent the coefficient matrix, $a_i$and b are known as nodes and weights respectively. For convenience the data is arranged in the given form called as Butcher table,

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

As

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{12} & a_{22} & \cdots & a_{2s} \\ . & . & . & \cdots & . \\ \hline & b_1 & b_2 & \cdots & b_s \end{array}$$

2.1 Kepler problem

Simplest problem is the two-body problem named as Kepler's problem associated with orbital movement of the planets. It describes the movement of one body orbiting another. The mathematical equation governing the motion are

$$x_1{}' = y_1 \,,$$
$$x_2{}' = y_2 \,,$$
$$y_1{}' = \frac{x_1}{(x_1{}^2 + x_2{}^2)^{3/2}} \,,$$
$$y_2{}' = \frac{x_2}{(x_1{}^2 + x_2{}^2)^{3/2}}$$

Energy is defined as

$$H = \frac{y_1{}^2 + y_2{}^2}{2} - \frac{1}{\sqrt{x_1{}^2 + x_2{}^2}}$$

2.2 Implicit Form of Runge—Kutta Methods

A simple form of Implicit R-K method can be derived from Guass–Legendre by assuming the nodal points as zeroes of the shifted Legendre Polynomial $Q(y)$ at $0 \leq y \leq 1$, defined as

$$Q(y) = \frac{s!}{2s} \sum_{r=0}^{s} (-1)^{s-r} \binom{s}{r} \binom{s+r}{r} y^r$$

For $s = 1$, $Q(y) = -\frac{1}{2} + y$, which has root $c = \frac{1}{2}$.

So, the second order Implicit R-K method is written as

$$\begin{array}{c|cc} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

By taking the value of s to be, $Q(y) = y^2 - y + \frac{1}{6}$, which has roots $c = \frac{1}{2} - \frac{\sqrt{3}}{6}$, $\frac{1}{2} + \frac{\sqrt{3}}{6}$

And we obtain the fourth order Implicit R-K method. Its Butcher tableau is

$$\begin{array}{c|cc} \frac{1}{2} - \frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \hline \frac{1}{2} + \frac{\sqrt{3}}{6} & \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

2.3 Symplecticity of Runge--Kutta Methods

Hamiltonian equations have a remarkable property conservation of energy and symplectic flow [12]. For the numerical solution of Hamiltonian system these methods must preserving these properties. So, we require R-K methods to be symplectic. For the symplecticity, R-K methods must satisfy the following condition called as symplectic R-K methods.

$$b_i a_{ij} + b_j a_{ji} - b_i b_j = 0 \,.$$

3 General Linear Methods

A simple numerical method with the property of both multistage and multi-value is consider as general linear methods [8]. Most important property of this method is the vectors computed in the start of the step and the derivatives are also evaluated at each step. Each Yi is a linear combination of outputs $y_i^{[n]}$ along $f(Y_i^{[n]})$ (the derivatives) [11].

The general form of GLM is

$$\mathrm{Y} = hAf(Y^{\,n}) + Uy^{[n-1]}$$

$$y^{[n]} = hBf(Y^{\,n}) + V\,y^{[n-1]}$$

General linear methods commonly reduced in of R-K. The matrix of coefficient A is similar in both while taking the transpose of vector of R-K we have matrix B [1].

3.1 Symplecticity of General Linear Methods

Most important property of Symplecticity for the Hamiltonian system has preserved by R-K methods. Normally the methods having multistep and multivalued nature that cannot possess the symplectic property, unless one can receive a single current value from the previous step, treated as the initial guess [19, 20].

The symplecticy means that the inner product is same for both initial point and later point values [17]. So that the symplect general linear method of fourth order defined for the preservation of energy path. To facilitate the multiplicity of general linear methods, consider a matrix along with its norm as L such as

$$\|y\|_L{}^2 = \langle y\,, y\rangle_L = \sum_{i,j=1}^{r} l_{ij}\,\langle y_i\,, y_j\rangle$$

Moreover, a diagonal matrix D is required as

$$\left\langle y^{[n]}\,, y^{[n]}\right\rangle_L = \left\langle y^{[n-1]}\,, y^{[n-1]}\right\rangle_L + 2h \sum D_i\langle Y_i\ \,, F_i\rangle$$

Since the behavior is conservative, one should satisfy

$$2h\,\sum D_i\langle Y_i\ \,, F_i\rangle = 0$$

For general linear methods, initial value is obtained by the initial guess while the remaining are taken by considering method P and N method [6,7].

$$\begin{array}{c|c} A & U \end{array}$$

$$\begin{array}{c|c} B & V \end{array}$$

Is

$$\begin{array}{cc|cc} \frac{3\pm\sqrt{3}}{6} & 0 & 1 & -\frac{3\pm 2\sqrt{3}}{3} \\ \mp\frac{\sqrt{3}}{3} & \frac{3\pm\sqrt{3}}{6} & 1 & \frac{3\pm 2\sqrt{3}}{3} \\ \hline \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & -1 \end{array}$$

With Starting numerically as

$$\begin{array}{cc|c} \frac{3\pm\sqrt{3}}{6} & 0 & 1 \\ -\frac{3\pm\sqrt{3}}{3} & \frac{3\pm\sqrt{3}}{6} & 1 \\ \hline 0 & 0 & 1 \\ \pm\frac{\sqrt{3}-1}{8} & -\frac{\sqrt{3}\mp 1}{6} & 0 \end{array}$$

It has observed that both approaches contain similar number with opposite sign of $\sqrt{3}$. The methods P and N create the parasitic effect for the solution which can be controlled by providing the coefficient of parasitism as

$$\upsilon = \frac{1 \pm 2\sqrt{3}}{3}$$

4 Method

The symplectic R-K method is of implicit form, we have used it for the solution of Hamiltonian equation. We have applied the symplectic fourth order RK method and symplectic general linear methods using methods P and N and their composition. These methods have been implemented over two-body, five-body and nine-body planetary motion for their energy conservation.

4.1 Five body problem

Jovian problem represents the orbital movement of the Sun along four planets that are Jupiter, Saturn, Uranus and Neptune, taking these as point masses. Its governing motion's equation is defined as

$$r_i'' = \sum_{j=1, j\neq i}^{5} \frac{\upsilon_j (r_j(x) - r_i(x))}{\left\| r_j(x) - r_i(x) \right\|_2^{\,3}}.$$

Where the $\|.\|_2$ denotes the L_2 norm, υ_j is the product of gravitational constant and mass of defined body.

Energy is defined by

$$H = \frac{1}{2} m_i r_i' r_i' - G \sum_{j \neq i} \frac{m_i m_j}{\left\| r_j(x) - r_i(x) \right\|_2} \ , \qquad i = 1, 2, \cdots, 5\ , \quad j = 1, 2, \cdots, 5\ .$$

4.2 HRC problem

The Helin-Roman-Crockett (HRC) problem represents a comet having multiple close approaches with Jupiter. With additive position of comet in five body equation of motion we get the HRC governing equation as

$$r_6'' = \sum_{j=1}^{5} \frac{\upsilon_j (r_j(x) - r_6(x))}{\left\| r_j(x) - r_6(x) \right\|_2^3} \ , \qquad j \neq i$$

4.3 9-Planets problem

This problem defines the Sun's orbital motion along all planets. The equation governed by considering the number of bodies as ten in the five-body defined in the Jovian, we have the nine-body equation of motion.

5 Results and Discussion

Lot of work has been done for Runge-Kutta along multistep methods. Also, there exist many excellent symplectic integrators among Runge-Kutta methods to discuss the conservation of the movement. The multivalued nature of general linear methods leads towards the parasitic solutions.

The construction of a method with four stages. These techniques are implemented to use various implementations for Hamiltonian and structure preserving systems.

- We studied the conserved path of energy of Kepler, Jovian and nine body planetary bodies, by choosing *eccentricity* values as 0.1 and 0.5 for n = 10^6 with h 0.01. The results of R-K and general linear methods along their compositions are as shown in Figures 1, 2, 3 and 4 respectively for the Kepler problem. We have fixed the step size and the number of iterations for each case in each method.
- For Jovian problem their results of R-K and general linear methods along their compositions are as shown in Figures 5, 6, 7 and 8 respectively. As the comet mass is negligible so similar results are obtained as of five body results.
- The results of R-K and general linear methods along their compositions for the Nine body problem are shown in Figures 9 and 10 respectively. Same results for N and P method are obtained in this case as the trajectory path is same with additional number of planets and energy is conserved.

Energy error is measured in each case by takin the tolerance level as 10^6. In all experiments we have used the constant step size and MATLAB software have been used for the calculations.

6 Figures and Tables

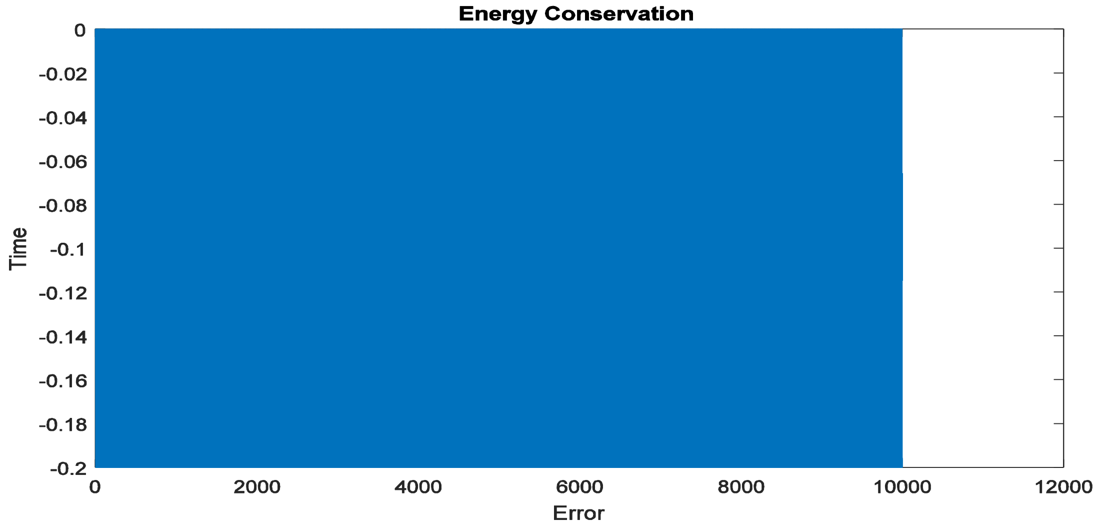

Figure 1. Energy Error of the Kepler problem with e=0.1 by R-K

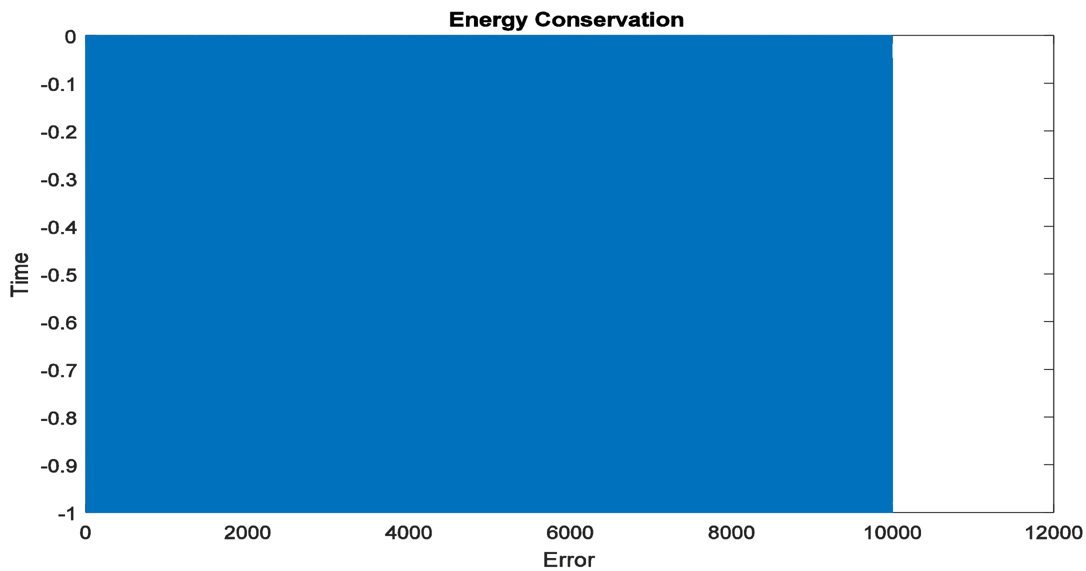

Figure 2. Energy Error of the Kepler problem with e=0.5 by R-K

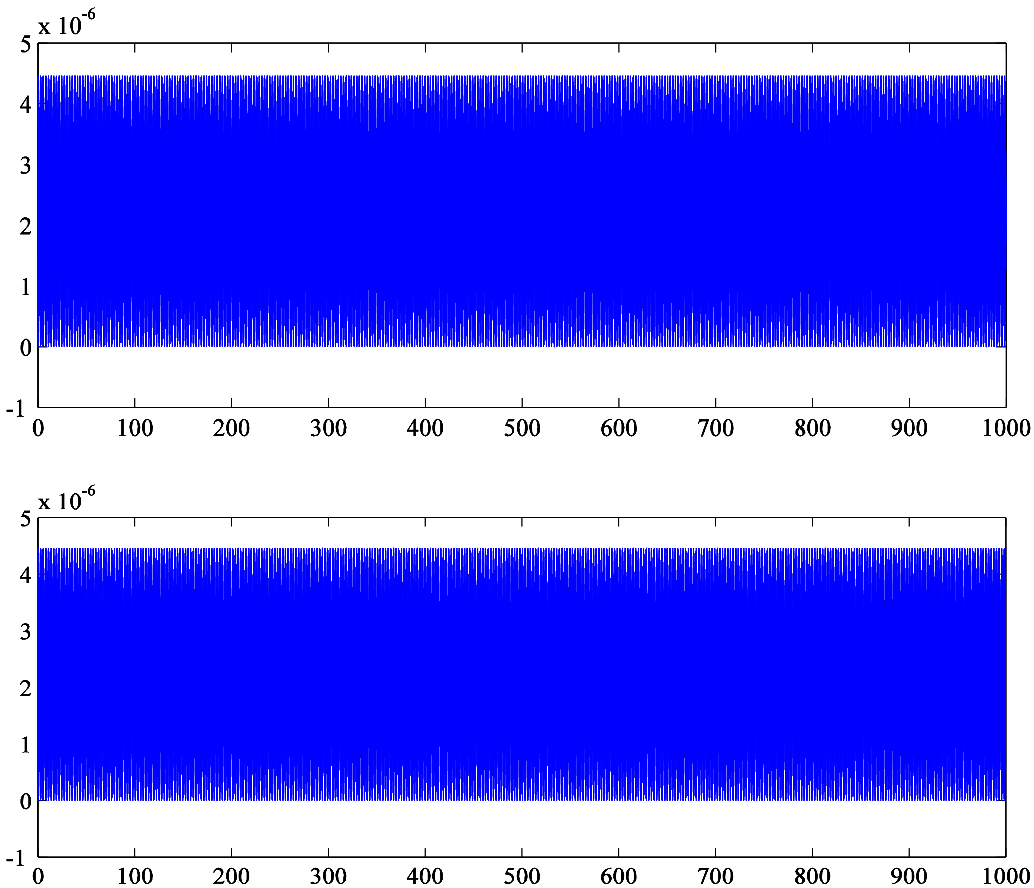

Figure 3. Energy Error of the Kepler problem by N method

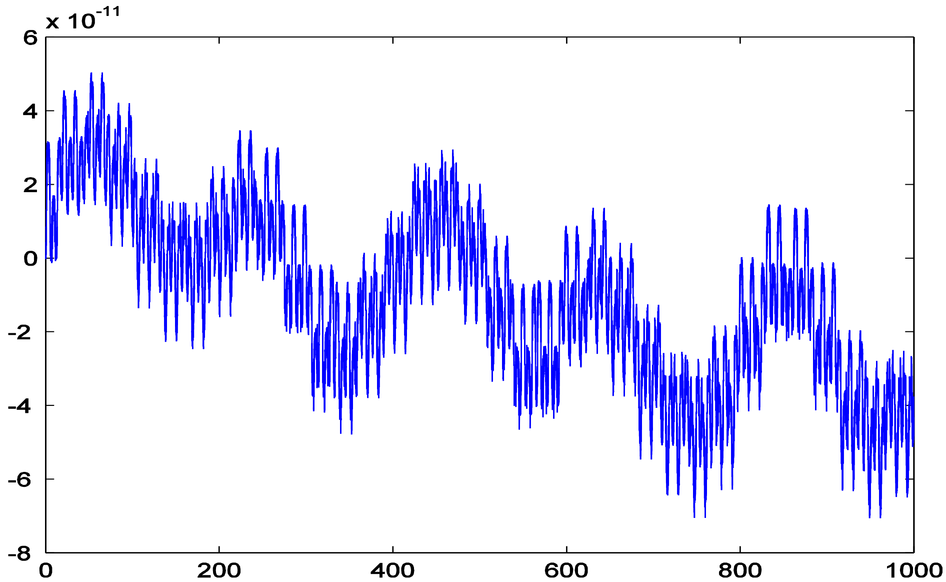

Figure 4. Energy Error of the Kepler problem by Composition

For the experiment, we take step-size of 2pi/10n where n=1,2,3,4, to study the conservative energy by symplectic R-K and fourth order general linear method. We achieved reasonable energy conservation as defined from a G-symplectic method which is parasitic-free and is shown in Fig.4. Regular oscillations in Fig.4 are indicated the fact that error was so small that can be neglected easily.

Furthermore, we have used different step-size of 2pi/10n where n = 1,2,3,4, for Kepler's problem and obtained the similar behavior for position and energy for the growth of error. 1 and 2 figures show the growth of error for position and energy by taking step-size of 2pi/10n where n = 1,2,3,4, for symplectic R-K method. Among the methods tested, composition of general linear method exhibited very small amount of energy error in terms of integration.

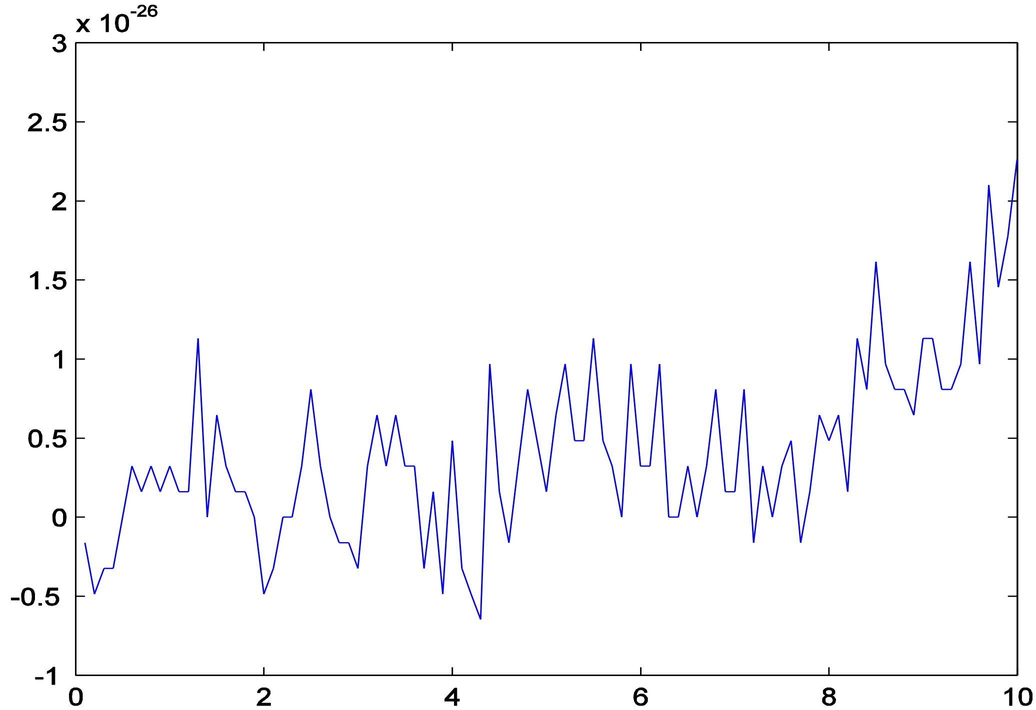

Figure 5. Energy Error of the Jovian problem by R-K

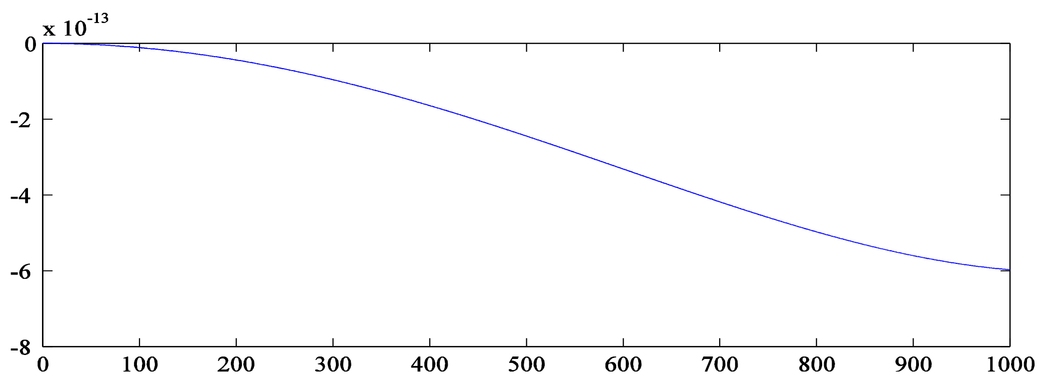

Figure 6. Energy Error of the Jovian problem by N

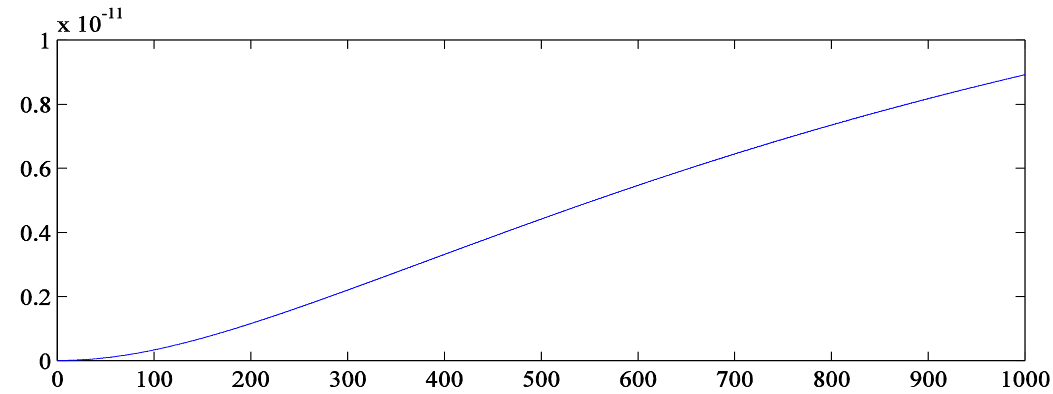

Figure 7. Energy Error of the Jovian problem by P

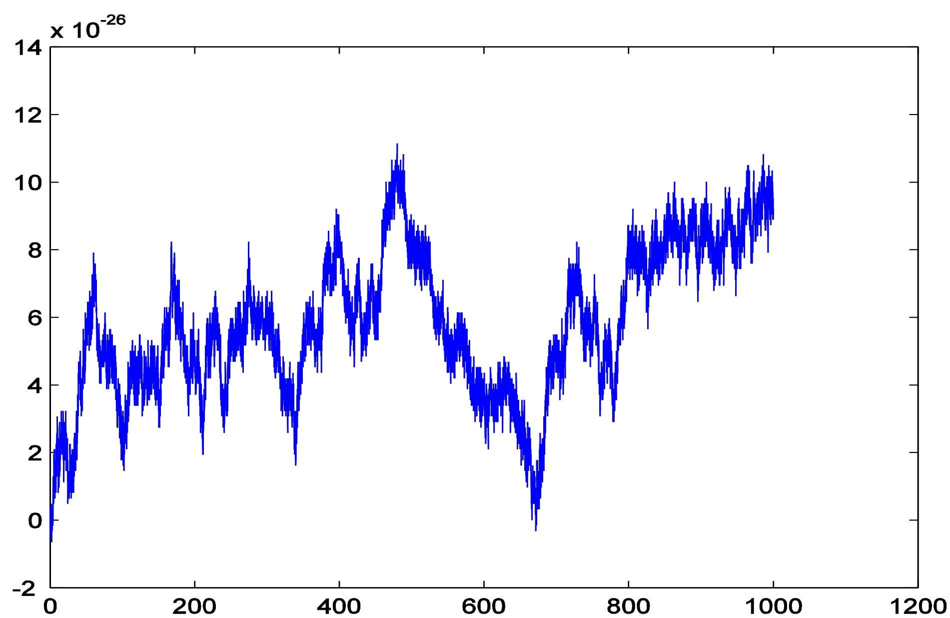

Figure 8. Energy Error of the Jovian by Composition

By investigating the Jovian problem for a 1000 year using the step-size of 50 &100, we obtained the same results. For the Jovian problem, figures 5,6,7 and 8 shows that the growth in error for the planetary positions and energy using step-size of 50 & 100 in defined methods. The magnitude of error follows the same pattern as was in Kepler's problem. However, symplectic R-K, and composition of fourth order general linear methods is having the smaller error in energy.

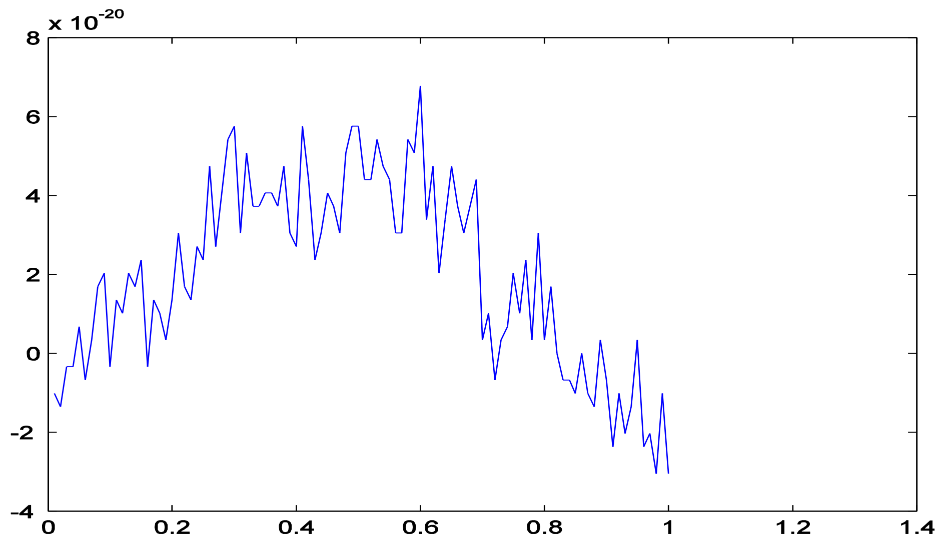

Figure 9. Energy Error of the Nine body by R-K

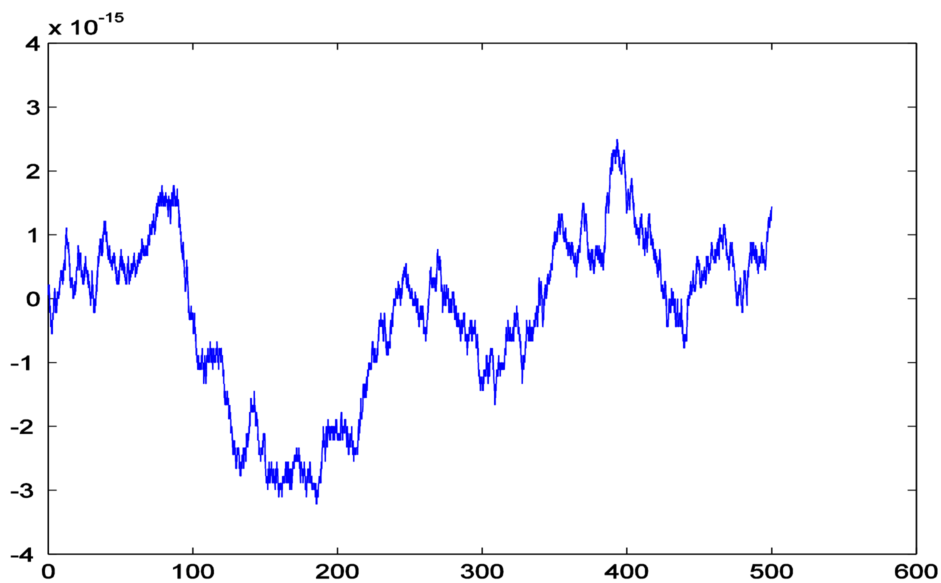

Figure 10. Energy Error of the Nine body by Composition

By investigating the nine-planet problem having the step-size of 1day, that is considering 50 times less for the orbital period of the Jovian problem. The error in energy found following the same procedure as was discussed in the Jovian problem defined in figures 9 and 10, which sows the conservation of the symplectic R-K methods and the composition of general linear method The less error in energy shows the conservation of the applied methods.

7 Conclusion

Numerically we integrate the equations by symplectic Runge-Kutta method and symplectic General Linear method along the composite general linear method to investigate the energy preservation of the defined integrals. It has been observed that these numerical methods not only provide the good numerical results also the qualitative preservation is obtained. We discuss the implicit R-K method and general linear methods along their composition in terms of symplecticity and applied them to the Hamiltonian systems separable. The energy is well conserved in each case. We deal with two body, five body and nine body problems defined by Hamiltonian systems for the solar dynamics. These methods of symplecticity and G-symplecticity are very useful for the solar systems and good energy conservations are observed.

References

[1] Leveque, Randall J. *Finite difference methods for ordinary and partial differential equations: steady-state and time-dependent problems*. Society for Industrial and Applied Mathematics, 2007.

[2] DE, NUMERICAL SOLUTIONS OF ORDINARY. "Numerical Solutions of Ordinary Differential Euations Numerical Analysis Numerical Methods-Solution of Equations, Interpolation, Numerical Integration, Numerical Solution of ODE Euler's Method Differential Equations, Examples, Numerical Methods, Calculus."J. C. Butcher. Implicit Runge-Kutta process.Math, Comp. 8 : 50 − 64, 1964,.

[3] Butcher, John Charles. *The numerical analysis of ordinary differential equations: Runge-Kutta and general linear methods*. Wiley-Interscience, 1987.

[4] Butcher, John C. "Coefficients for the study of Runge-Kutta integration processes." *Journal of the Australian Mathematical Society* 3, no. 2 (1963): 185-201.J. C. Butcher. General linear methods: A survey, 1985.

[5] Butcher, John C., Yousaf Habib, Adrian T. Hill, and Terence JT Norton. "The control of parasitism in G-symplectic methods." *SIAM Journal on Numerical Analysis* 52, no. 5 (2014): 2440-2465.

[6] Dahlquist, Germund G. "A special stability problem for linear multistep methods." *BIT Numerical Mathematics* 3, no. 1 (1963): 27-43.

[7] Mustafa, Rafay. "Symplectic Effect for the Numerical Solution of Conservative Systems." *International Journal of Artificial Intelligence & Mathematical Sciences* 1, no. 1 (2022): 46-59.

[8] Hairer, Ernst, Christian Lubich, and Gerhard Wanner. "Structure-preserving algorithms for ordinary differential equations." *Geometric numerical integration* 31 (2006).

[9] Mackenzei, J. "A. Iserles A first course in the numerical analysis of differential equations (Cambridge University Press, Cambridge, 1996), xvi+ 378pp., 0 521 55655 4 (paperback)£ 19.95 (US74. 95)." *Proceedings of the Edinburgh Mathematical Society* 41, no. 1 (1998): 211-211.

[10] Prince, Peter J., and John R. Dormand. "High order embedded Runge-Kutta formulae." *Journal of computational and applied mathematics* 7, no. 1 (1981): 67-75.

[11] Mustafa, Rafay. "Symplectic Effect for the Numerical Solution of Conservative Systems." *International Journal of Artificial Intelligence & Mathematical Sciences* 1, no. 1 (2022): 46-59.

[12] Sanz-Serna, Jesus M. "Symplectic integrators for Hamiltonian problems: an overview." *Acta numerica* 1 (1992): 243-286.

[13] Sanz-Serna, Jesus-Maria, and Mari-Paz Calvo. *Numerical hamiltonian problems*. Vol. 7. Courier Dover Publications, 2018.Sanz-Serna, J. M., and L. Abia. "Order conditions for canonical Runge–Kutta schemes." *SIAM Journal on Numerical Analysis* 28, no. 4 (1991): 1081-1096.

[14] Bogfjellmo, Geir, Charles H. Curry, and Dominique Manchon. "Rooted trees, non-rooted trees and Hamiltonian B-series." *arxiv preprint arxiv:1406.0821* (2014).

[15] Mustafa, Rafay. "Symplectic Effect for the Numerical Solution of Conservative Systems." *International Journal of Artificial Intelligence & Mathematical Sciences* 1, no. 1 (2022): 46-59.

[16] Tang, Yi-Fa. "The symplecticity of multi-step methods." *Computers & Mathematics with Applications* 25, no. 3 (1993): 83-90.

[17] Wright, William Matthew. "General linear methods with inherent Runge-Kutta stability." phd diss., researchspace@ Auckland, 2002.

[18] Habib, Yousaf. "Long-term behaviour of G-symplectic methods." phd diss., researchspace@ Auckland, 2010.

[19] Habib, Y., R. Mustafa, and M. A. Qureshi. "G-Symplectic Integration of Many Body Problems." *Bulletin of the Iranian Mathematical Society* 44 (2018): 937-954.

[20] Ahmad, Junaid, Yousaf Habib, Shafiq ur Rehman, Azqa Arif, Saba Shafiq, and Muhammad Younas. "Symplectic effective order numerical methods for separable Hamiltonian systems." *Symmetry* 11, no. 2 (2019): 142.