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Generalized Inequalities of Hermite-Hadamard Like for $(m_1, m_2)-$Convex Function Via Hölder & Power-Mean Inequalities and Implementations

[1*]Faraz Mehmood

faraz.mehmood@duet.edu.pk

[1]Department of Mathematics, Dawood University of Engineering and Technology, New M. A. Jinnah Road, Karachi, Pakistan.

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Abstract

In the present article, we prove the generalized Hermite-Hadamard (H-H) like inequalities for $(m_1, m_2)-$convex (conv.) via Hölder & Power-mean inequalities and implementations for theory of probability (prob.) & numerical integration are deduced. Some consequences of several published articles would be captured as especial cases. Moreover, we deduce few especial cases of $(m_1, m_2)-$conv. function (func.).

Keywords: Convex func., H-H inequality (inequal.), Power-mean inequal., Hölder inequal., Numerical integration, Prob. density func.

*Email: faraz.mehmood@duet.edu.pk

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1. INTRODUCTION

We must generalize the idea of convex functions to generalize Ostrowski's inequality. In this way, we may quickly identify the generalizations and specific instances of the inequal. We recollect several definitions from literature [2,3,6,7] for variety of convex functions.

Definition 1.1. A func. g : K $\subseteq (-\infty, \infty) \to (-\infty, \infty)$ is known as conv., if

$$g(\vartheta\, \mathcal{v} + (1-\vartheta)\, \mathcal{w}) \le \vartheta\, g(\mathcal{v}) + (1-\vartheta) g(\mathcal{w}), \tag{1.1}$$

$\forall\, \mathcal{w}, \mathcal{v} \in K,\ \vartheta \in [0,1]$.

We remind term of $M-$conv. func. (see [4]).

Definition 1.2. Let $M \in [0,1]$. A func. g : K $\subseteq [0,\infty) \rightarrow (-\infty,\infty)$ is known as $M-$conv., if

$$g(\vartheta \, v + M\,(1-\vartheta)\, w) \leq \vartheta \, g(v) + M\,(1-\vartheta) g(w),$$

$\forall\, w, v \in$ K, $\vartheta \in [0,1]$.

Remark 1.4. The terms of standard conv. func. & star-shaped func. are obtained if choose M =1, M =0 in the above inequality respectively.

We remind term of $(m_1, m_2)-$conv. func. (see [4]).

Definition 1.3. Let $(m_1, m_2) \in (0,1]^2$. A func. g : K $\subseteq [0,\infty) \rightarrow (-\infty,\infty)$ is known as $(m_1, m_2)-$conv., if

$$g(m_1\vartheta \, v + m_2\,(1-\vartheta)\, w) \leq m_1\vartheta \, g(v) + m_2\,(1-\vartheta) g(w), \tag{1.2}$$

$\forall\, w, v \in$ K, $\vartheta \in [0,1]$.

In practically all scientific fields, inequalities have a major impact. Our primary target is on H–H like inequalities, despite the discipline's enormous scope.

The convexity theory is closely connected to inequality's theory. Many well-known Inequalities in literature consequence directly from applying conv. functions. The H–H equation is a notable equation for conv. funct. that has been thoroughly researched in recent decades. It is discovered independently by Hadamard & Hermite, and it is stated as follows: Any func. g : K $\subseteq (-\infty,\infty) \rightarrow (-\infty,\infty)$ be conv., $m, \ell \in$ K with $m > \ell$, If & only if

$$g\left(\frac{\ell+m}{2}\right) \leq \frac{1}{m-\ell}\int_{\ell}^{m} g(v)dv \leq \frac{g(\ell)+g(m)}{2} \tag{1.3}$$

this is known as H-H inequal. Equation (1.3) has become a crucial pillar in the area of prob. & optimization. Additionally, numerous researchers have refined or generalized equation (1.3) for conv., s- conv., quasi- conv., & various other varieties of functions.

In [1], the following consequence had derived by Agarwal & Dragomir, which includes the H-H like integral inequal.

Proposition 1.4. Let g : K0 $\subseteq (-\infty,\infty) \rightarrow (-\infty,\infty)$ be differentiable mapping (diff. map.) in the Interior Ko of K, here $m, \ell \in$ Ko along $m > \ell$. If |g'| is conv. in interval $[\ell, m]$. Then the below equation holds

$$\left|\frac{g(\ell)+g(m)}{2} - \frac{1}{m-\ell}\int_{\ell}^{m} g(u)du\right| \leq \frac{(m-\ell)(|g\prime(\ell)|+|g\prime(m)|)}{8} \tag{1.4}$$

For additional recent consequences on H-H like inequalities involving various classes of conv. functions, refer to [5,8,9]. Kavurmaci et al. [5] established few novel inequalities of H-H like for conv. functions & implementations by utilizing

The primary goal of the article is to generalize few H-H like inequalities to $(m_1, m_2)-$conv. func. by employing the Hölder & Powermean inequalities. The implementations also encompass areas such as prob. & numerical integration. We will capture some findings of various articles [1,5] and also examine especial cases of $(m_1, m_2)-$conv. func.

2. GENERALIZED HERMITE-HADAMARD LIKE INEQUALITIES FOR $(m_1, m_2)-$CONVEX FUNCTION

Regarding proof of our primary findings, below Lemma (see [5]) is required.

Lemma 2.1. Suppose g : K $\subseteq (-\infty,\infty) \rightarrow (-\infty,\infty)$ is diff. map. in interval K0 $\subset (-\infty,\infty)$ here m_1, $m_2 m$, $m_2\ell \in$ K along $m_2 m > m_2\ell$, if g' $\in$ L[$m_2\ell$,$m_2 m$], then

$$\frac{(m_2 m - m_1 v)g(m_2 m) + (m_1 v - m_2 \ell)g(m_2\ell)}{m-\ell} - \frac{1}{m-\ell}\int_{m_2\ell}^{m_2 m} g(u)du$$

$$= \frac{(m_1 v - m_2\ell)^2}{m-\ell}\int_0^1 (1-\vartheta)g'(m_1\vartheta v + m_2(1-\vartheta)\ell)d\vartheta + \frac{(m_2 m - m_1 v)^2}{m-\ell}\int_0^1 (1-\vartheta)g'(m_1\vartheta v + m_2(1-\vartheta)m)d\vartheta.$$

Proof. We obtain the desired consequence by employing likewise techniques of proof of Lemma 1 of [5].

Remark 2.1. We recapture Lemma 1 of [5] if choose m_1, m_2 = 1.

The below consequences may be derived by employing lemma 2.1

Theorem 2.2. Suppose g : K $\subseteq (-\infty,\infty) \rightarrow (-\infty,\infty)$ is diff. map. in interval K0 $\subset (-\infty,\infty)$ ε $g' \epsilon$ $L[m_2\ell, m_2 m]$, here m_1, $m_2 m$, $m_2\ell \in$ K along $m_2 m > m_2\ell$, if |g'| is $(m_1, m_2)-$conv. on $[m_2\ell, m_2 m]$, then

$$\left|\frac{(m_2 m - m_1 v)g(m_2 m) + (m_1 v - m_2 \ell)g(m_2\ell)}{m-\ell} - \frac{1}{m-\ell}\int_{m_2\ell}^{m_2 m} g(u)du\right|$$

$$\leq \frac{(m_1 v - m_2\ell)^2}{m-\ell}\left[\frac{m_1|g'(v)|+2m_2|g'(\ell)|}{6}\right] + \frac{(m_2 m - m_1 v)^2}{m-\ell}\left[\frac{m_1|g'(v)|+2m_2|g'(m)|}{6}\right].$$

For every $g\epsilon[m_2\ell, m_2 m]$

Proof. Utilizing lemma 2.1 & taking Modulus, then

$$\left|\frac{(m_2 m - m_1 v)g(m_2 m) + (m_1 v - m_2 \ell)g(m_2\ell)}{m-\ell} - \frac{1}{m-\ell}\int_{m_2\ell}^{m_2 m} g(u)du\right|$$

$$\leq \frac{(m_1 v - m_2\ell)^2}{m-\ell}\int_0^1 (1-\vartheta)|g'(m_1\vartheta v + m_2(1-\vartheta)\ell)|d\vartheta + \frac{(m_2 m - m_1 v)^2}{m-\ell}\int_0^1 (1-\vartheta)|g'(m_1\vartheta v + m_2(1-\vartheta)m)|d\vartheta$$

$\because$ |g' | is $(m_1, m_2)-$conv., then

$$\left|\frac{(m_2 m - m_1 v)g(m_2 m) + (m_1 v - m_2 \ell)g(m_2\ell)}{m-\ell} - \frac{1}{m-\ell}\int_{m_2\ell}^{m_2 m} g(u)du\right|$$

$\leq \frac{(m_1 v - m_2 \ell)^2}{m-\ell} \int_0^1 (1-\vartheta)[m_1\vartheta|g'(v)| + m_2(1-\vartheta)|g'(\ell)|]\, d\vartheta$

$+ \frac{(m_2 m - m_1 v)^2}{m-\ell} \int_0^1 (1-\vartheta)\,[m_1\vartheta|g'(v)| + m_2(1-\vartheta)|g'(m)|] d\vartheta$

$= \frac{(m_1 v - m_2 \ell)^2}{m-\ell}\left[\frac{m_1|g'(v)|+2m_2|g'(\ell)|}{6}\right] + \frac{(m_2 m - m_1 v)^2}{m-\ell}\left[\frac{m_1|g'(v)|+2m_2|g'(m)|}{6}\right]$

thus completing the proof.

Remark 2.3. The above result is also obtained for term of star-shaped func. if choose m_1= 1, m_2=0 in Theorem 2.2.

Remark 2.4. We attain the Theorem 4 of [5] If choose m_1, m_2 = 1 in Theorem 2.2.

Corollary 2.5. In Theorem 2.2, choosing $v = \frac{m_2\ell + m_2 m}{2}$ we get

$$\left|\frac{(2m_2 m - m_1 m_2 \ell - m_1 m_2 m) g(m_2 m) + (m_1 m_2 \ell + m_1 m_2 m - 2m_2\ell) g(m_2\ell)}{2(m-\ell)} - \frac{1}{m-\ell}\int_{m_2\ell}^{m_2 m} g(u)du\right| \leq$$

$$\frac{(m_1 m_2 \ell + m_1 m_2 m - 2m_2\ell)^2}{m-\ell}\left[\frac{m_1\left|g'\left(\frac{m_2\ell+m_2m}{2}\right)\right|+2m_2|g'(\ell)|}{24}\right] + \frac{(2m_2 m - m_1 m_2 \ell - m_1 m_2 m)^2}{m-\ell}\left[\frac{m_1\left|g'\left(\frac{m_2\ell+m_2m}{2}\right)\right|+2m_2|g'(m)|}{24}\right]$$

Remark 2.6. Few Remarks regarding Corollary 2.5 are below as especial cases.

By utilizing the convexity property of |g'| in Corollary 2.5, we get established equation (1.4) (capture theorem 2.2 of article [1]).

The above result is also obtained for term of star-shaped func. if choose m_1= 1, m_2=0 in Corollary 2.5.

Remark 2.7. We capture the Corollary 2 of article [5] If choose m_1, m_2 = 1 in Corollary 2.5.

Theorem 2.8. Suppose g : K $\subseteq (-\infty,\infty) \to (-\infty,\infty)$ is diff. map. in K0 $\subset (-\infty,\infty)$ ε $g' \epsilon L[m_2\ell, m_2 m]$, here m_1, $m_2 m$, $m_2\ell \in$ K along $m_2 m > m_2\ell$. If $|g'|^{\frac{p}{p-1}}$ is $(m_1, m_2)-$conv. in the interval $[m_2\ell, m_2 m]$ & for some fixed 1 < q. Then

$$\left|\frac{(m_2 m - m_1 v) g(m_2 m) + (m_1 v - m_2\ell) g(m_2\ell)}{m-\ell} - \frac{1}{m-\ell}\int_{m_2\ell}^{m_2 m} g(u)du\right|$$

$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[\frac{(m_1 v - m_2\ell)^2}{m-\ell}(m_1|g'(v)|^q + m_2|g'(\ell)|^q)^{\frac{1}{q}} + \frac{(m_2 m - m_1 v)^2}{m-\ell}(m_1|g'(v)|^q + m_2|g'(m)|^q)^{\frac{1}{q}}\right]$$

for every g $\in [m_2\ell, m_2 m]$ and q=$\frac{p}{p-1}$.

Proof. Utilizing Lemma 2.1 & $(m_1, m_2)-$convexity of |g'| & then implementing the widely recognized Hölder inequality, we get

$$\left|\frac{(m_2 m - m_1 v) g(m_2 m) + (m_1 v - m_2\ell) g(m_2\ell)}{m-\ell} - \frac{1}{m-\ell}\int_{m_2\ell}^{m_2 m} g(u)du\right|$$

$$\leq \frac{(m_1 \mathcal{v} - m_2 \ell)^2}{m - \ell} \int_0^1 (1-\vartheta)|g'(m_1\vartheta\mathcal{v} + m_2(1-\vartheta)\ell)|d\vartheta$$

$$+ \frac{(m_2 m - m_1 \mathcal{v})^2}{m - \ell} \int_0^1 (1-\vartheta)|g'(m_1\vartheta\mathcal{v} + m_2(1-\vartheta)m)|d\vartheta$$

$$\leq \frac{(m_1 \mathcal{v} - m_2 \ell)^2}{m - \ell} \int_0^1 (1-\vartheta)[m_1\vartheta|g'(\mathcal{v})| + m_2(1-\vartheta)|g'(\ell)|]\, d\vartheta$$

$$+ \frac{(m_2 m - m_1 \mathcal{v})^2}{m - \ell} \int_0^1 (1-\vartheta)\, [m_1\vartheta|g'(\mathcal{v})| + m_2(1-\vartheta)|g'(m)|]d\vartheta$$

$$\leq \frac{(m_1 \mathcal{v} - m_2 \ell)^2}{m - \ell} \left(\int_0^1 (1-\vartheta)^p d\vartheta\right)^{\frac{1}{p}} \left[\int_0^1 (m_1\vartheta|g'(\mathcal{v})| + m_2(1-\vartheta)|g'(\ell)|)^q d\vartheta\right]^{\frac{1}{p}}$$

$$+ \frac{(m_2 m - m_1 \mathcal{v})^2}{m - \ell} \left(\int_0^1 (1-\vartheta)^p d\vartheta\right)^{\frac{1}{p}} \left[\int_0^1 (m_1\vartheta|g'(\mathcal{v})| + m_2(1-\vartheta)|g'(m)|)^q d\vartheta\right]^{\frac{1}{p}}$$

$$\leq \frac{(m_1 \mathcal{v} - m_2 \ell)^2}{m - \ell} \left(\int_0^1 (1-\vartheta)^p d\vartheta\right)^{\frac{1}{p}} \left[m_1 \int_0^1 \vartheta|g'(\mathcal{v})|^q d\vartheta + m_2 \int_0^1 (1-\vartheta)|g'(\ell)|^q d\vartheta\right]^{\frac{1}{p}}$$

$$+ \frac{(m_2 m - m_1 \mathcal{v})^2}{m - \ell} \left(\int_0^1 (1-\vartheta)^p d\vartheta\right)^{\frac{1}{p}} \left[m_1 \int_0^1 \vartheta|g'(\mathcal{v})|^q d\vartheta + m_2 \int_0^1 (1-\vartheta)|g'(m)|^q d\vartheta\right]^{\frac{1}{p}}$$

$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2}\right)^{\frac{1}{q}} \left[\frac{(m_1 \mathcal{v} - m_2 \ell)^2}{m - \ell} (m_1|g'(\mathcal{v})|^q + m_2|g'(\ell)|^q)^{\frac{1}{q}} + \frac{(m_2 m - m_1 \mathcal{v})^2}{m - \ell} (m_1|g'(\mathcal{v})|^q + m_2|g'(m)|^q)^{\frac{1}{q}}\right]$$

Remark 2.9. The above result is also obtained for term of star-shaped func. if choose m_1= 1, m_2=0 in Theorem 2.8.

Remark 2.10. We attain the Theorem 5 of [5] If choose m_1, m_2 = 1 in Theorem 2.8.

Corollary 2.11. In Theorem 2.8, choosing $\mathcal{v} = \frac{m_2\ell + m_2 m}{2}$ we get

$$\left|\frac{(2m_2 m - m_1 m_2 \ell - m_1 m_2 m) g(m_2 m) + (m_1 m_2 \ell + m_1 m_2 m - 2m_2 \ell) g(m_2 \ell)}{2(m-\ell)} - \frac{1}{m-\ell}\int_{m_2\ell}^{m_2 m} g(u)du\right| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2}\right)^{\frac{1}{q}} \frac{1}{4(m-\ell)} \Bigg[(m_1 m_2 \ell + m_1 m_2 m - 2m_2\ell)^2 \left(m_2|g'(\ell)|^q + m_1\left|g'\left(\frac{m_2\ell+m_2 m}{2}\right)\right|^q\right)^{\frac{1}{q}} + (2m_2 m - m_1 m_2 \ell - m_1 m_2 m)^2 \left(m_2|g'(m)|^q + m_1\left|g'\left(\frac{m_2\ell+m_2 m}{2}\right)\right|^q\right)^{\frac{1}{q}}\Bigg]$$

Remark 2.12. The above result is also obtained for term of star-shaped func. if choose m_1= 1, m_2=0 in Corollary 2.11.

Remark 2.13. We capture the Corollary 3 of article [5] If choose m_1, m_2 = 1 in Corollary 2.11.

Theorem 2.14. Suppose g : K $\subseteq (-\infty,\infty) \to (-\infty,\infty)$ is diff. map. on K0 $\subset (-\infty,\infty)$ ε $g' \epsilon L[m_2\ell, m_2 m]$, here m_1,

$m_2 m$, $m_2 \ell \in$ K along $m_2 m > m_2 \ell$. If |g'|q is $(m_1, m_2)-$conv. on $[m_2\ell, m_2 m]$ & for some fixed $q \geq 1$. Then

$$\left|\frac{(m_2 m - m_1 v)g(m_2 m) + (m_1 v - m_2 \ell)g(m_2 \ell)}{m - \ell} - \frac{1}{m - \ell}\int_{m_2 \ell}^{m_2 m} g(u)du\right|$$

$$\leq \frac{1}{2\,(3)^{\frac{1}{q}}}\left[\frac{(m_1 v - m_2 \ell)^2}{m - \ell}\left(m_1|g'(v)|^q + 2m_2|g'(\ell)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\frac{(m_2 m - m_1 v)^2}{m - \ell}\left(m_1|g'(v)|^q + 2m_2|g'(m)|^q\right)^{\frac{1}{q}}\right]$$

for every $g \epsilon\ [m_2 \ell, m_2 m], q = \frac{p}{p-1}$

Proof. Consider 1 ≤q & Utilizing Lemma 2.1 & implementing the widely recognized power-mean inequality, we get

$$\left|\frac{(m_2 m - m_1 v)g(m_2 m) + (m_1 v - m_2 \ell)g(m_2 \ell)}{m - \ell} - \frac{1}{m - \ell}\int_{m_2 \ell}^{m_2 m} g(u)du\right|$$

$$\leq \frac{(m_1 v - m_2 \ell)^2}{m - \ell}\int_0^1 (1-\vartheta)|g'(m_1\vartheta v + m_2(1-\vartheta)\ell)|d\vartheta$$

$$+\frac{(m_2 m - m_1 v)^2}{m - \ell}\int_0^1 (1-\vartheta)|g'(m_1\vartheta v + m_2(1-\vartheta)m)|d\vartheta$$

$$\leq \frac{(m_1 v - m_2 \ell)^2}{m - \ell}\left(\int_0^1 (1-\vartheta)d\vartheta\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-\vartheta)|g'(m_1\vartheta v + m_2(1-\vartheta)\ell)|^q d\vartheta\right)^{\frac{1}{q}}$$

$$+\frac{(m_2 m - m_1 v)^2}{m - \ell}\left(\int_0^1 (1-\vartheta)d\vartheta\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-\vartheta)|g'(m_1\vartheta v + m_2(1-\vartheta)m)|^q d\vartheta\right)^{\frac{1}{q}}$$

∵ |g' | is $(m_1, m_2)-$conv. & first we are taking term

$$\int_0^1 (1-\vartheta)|g'(m_1\vartheta v + m_2(1-\vartheta)\ell)|^q d\vartheta$$

$$\leq \int_0^1 (1-\vartheta)[m_1\vartheta|g'(v)|^q + m_2(1-\vartheta)|g'(\ell)|^q]\, d\vartheta$$

$$= \frac{m_1|g'(v)|^q + 2m_2|g'(\ell)|^q}{6}$$

Analogously,

$$\int_0^1 (1-\vartheta)|g'(m_1\vartheta v + m_2(1-\vartheta)m)|^q d\vartheta \leq \frac{m_1|g'(v)|^q + 2m_2|g'(m)|^q}{6}.$$

We obtain the desired result by combining all above inequalities.

Remark 2.15. The above result is also obtained for term of star-shaped func. if choose m_1= 1, m_2=0 in Theorem 2.14.

Remark 2.16. We attain the Theorem 7 of article [5] If choose m_1, m_2 = 1 in The0rem 2.14.

Corollary 2.17. In Theorem 2.14, choosing $\upsilon = \frac{m_2\ell+m_2 m}{2}$ we get

$$\left|\frac{(2m_2 m-m_1m_2\ell-m_1m_2m)g(m_2m)+(m_1m_2\ell+m_1m_2m-2m_2\ell)g(m_2\ell)}{2(m-\ell)} - \frac{1}{m-\ell}\int_{m_2\ell}^{m_2m} g(u)du\right|$$

$$\leq \frac{1}{8\,(3)^{\frac{1}{q}}\,(m-\ell)}\left[(m_1m_2\ell+m_1m_2m-2m_2\ell)^2\left(m_1\left|g'\left(\frac{m_2\ell+m_2m}{2}\right)\right|^q + 2m_2|g'(\ell)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+(2m_2m-m_1m_2\ell-m_1m_2m)^2\left(m_1\left|g'\left(\frac{m_2\ell+m_2m}{2}\right)\right|^q + 2m_2|g'(m)|^q\right)^{\frac{1}{q}}\right].$$

Remark 2.18. The above result is also obtained for term of star-shaped func. if choose m_1= 1, m_2=0 in Corollary 2.17.

Remark 2.19. We capture the Corollary 4 of article [5] If choose m_1, m_2 = 1 in Corollary 2.17.

3. IMPLEMENTATION TO NUMERICAL INTEGRATION

3.1. The Trapezoidal Formula. Suppose d : $m_2\ell = m_2\theta_0 < m_2\theta_1 < \cdots < m_2\theta_n = m_2 m$ is divisi0n 0f interval $[m_2\ell, m_2m]$ & $h_i = \theta_{i+1} - \theta_i$, (i = o,1,2,⋯ ,n − 1) & consider the quadrature formula

$$\int_{m_2\ell}^{m_2m} g(u)du = Q(g,d) + R(g,d), \quad (3.1)$$

where

$$Q(g,d) = \sum_{i=0}^{n-1}((m_2\theta_{i+1} - m_1\upsilon)g(m_2\theta_{i+1}) + (m_1\upsilon - m_2\theta_i)g(m_2\theta_i))$$

for the trapez0idal version & R(g,d) represents the associated approximati0n error.

Theorem 3.1. With all the suppositions of Theorem 2.2, for every division d of the interval $[m_2\ell, m_2m]$. Then in equation (3.1), the trapezoidal error estimate satisfies:

$$|R(g,d)|$$

$$\leq \sum_{i=0}^{n-1}(m_1\upsilon - m_2\theta_i)^2\left[\frac{m_1|g'(\upsilon)| + 2m_2|g'(\theta_i)|}{6}\right] + \sum_{i=0}^{n-1}(m_2\theta_{i+1} - m_1\upsilon)^2\left[\frac{m_1|g'(\upsilon)| + 2m_2|g'(\theta_{i+1})|}{6}\right]$$

for every g ∈ $[m_2\ell, m_2m]$.

Proof. Utilizing theorem 2.2 on sub-interval $[m_2\theta_i , m_2\theta_{i+1}]$ here i = o,1,2 ,..., n − 1 of the division, we get

$$\left|\frac{(m_2\theta_{i+1} - m_1\upsilon)g(m_2\theta_{i+1}) + (m_1\upsilon - m_2\theta_i)g(m_2\theta_i)}{h_i} - \frac{1}{h_i}\int_{m_2\theta_i}^{m_2\theta_{i+1}} g(u)du\right|$$

$$\leq \frac{(m_1\upsilon-m_2\theta_i)^2}{h_i}\left[\frac{m_1|g'(\upsilon)|+2m_2|g'(\theta_i)|}{6}\right] + \frac{(m_2\theta_{i+1}-m_1\upsilon)^2}{h_i}\left[\frac{m_1|g'(\upsilon)|+2m_2|g'(\theta_{i+1})|}{6}\right]$$

Now we take summation over i in equation (3.1) from 0 to n − 1, then

$$\left|\int_{m_2\ell}^{m_2m} g(u)du - Q(g,d)\right| = \left|\sum_{i=0}^{n-1}\left[\int_{m_2\theta_i}^{m_2\theta_{i+1}} g(u)du - ((m_2\theta_{i+1} - m_1\upsilon)g(m_2\theta_{i+1}) + (m_1\upsilon - m_2\theta_i)g(m_2\theta_i))\right]\right|$$

$\leq \sum_{i=0}^{n-1} \left| \int_{m_2\theta_i}^{m_2\theta_{i+1}} g(u)du - ((m_2\theta_{i+1} - m_1 v)g(m_2\theta_{i+1}) + (m_1 v - m_2\theta_i)g(m_2\theta_i)) \right|$

$$\leq \sum_{i=0}^{n-1} (m_1 v - m_2\theta_i)^2 \left[\frac{m_1|g'(v)| + 2m_2|g'(\theta_i)|}{6}\right] + \sum_{i=0}^{n-1} (m_2\theta_{i+1} - m_1 v)^2 \left[\frac{m_1|g'(v)| + 2m_2|g'(\theta_{i+1})|}{6}\right]$$

which completes the proof.

Remark 3.2. We can also derive above similar findings for Theorem 2.14, Corollary 2.17, Corollary 2.11, Theorem 2.8 & Corollary 2.5.

Remark 3.3. Entire Remarks of 2nd section are also holds for Remark 3.2 & Theorem 3.1.

4. IMPLEMENTATION TO PROBABILITY THEORY

Suppose $\mathfrak{R}$ is a random variable taking values in the finite $[m_2\ell, m_2 m]$ with the prob. density func. g : $[m_2\ell, m_2 m] \rightarrow [0,1]$ & the cumulative distribution func. $\mathcal{G}(v) = P(\mathfrak{R} \leq v) = \int_{m_2\ell}^{m_2 m} g(u)du$.

Theorem 4.1. With all the assumptions 0f Theorem 2.2, then

$$\left| \frac{(m_2 m - m_1 v)\mathcal{G}(m_2 m) + (m_1 v - m_2\ell)\mathcal{G}(m_2\ell)}{m - \ell} - \frac{m_2 m - \mathcal{E}(\mathfrak{R})}{m - \ell} \right|$$

$$\leq \frac{(m_1 v - m_2\ell)^2}{m-\ell}\left[\frac{m_1|\mathcal{G}'(v)| + 2m_2|\mathcal{G}'(\ell)|}{6}\right] + \frac{(m_2 m - m_1 v)^2}{m-\ell}\left[\frac{m_1|\mathcal{G}'(v)| + 2m_2|\mathcal{G}'(m)|}{6}\right] \quad (4.1)$$

for every g ∈ $[m_2\ell, m_2 m]$. Here $\mathcal{E}(\mathfrak{R})$ is the expectation of $\mathfrak{R}$.

Proof. Choose g = $\mathcal{G}$, we attain (4.1), by implementing the below identity in Theorem 2.2.

$\mathcal{E}(\mathfrak{R}) = \int_{m_2\ell}^{m_2 m} u\mathcal{G}(u)du = m_2 m - \int_{m_2\ell}^{m_2 m} \mathcal{G}(u)du.$

$\because \mathcal{G}(m_2\ell) = 0$ & $\mathcal{G}(m_2 m) = 1$.

Remark 4.3. Entire Remarks of 2nd section are also hold for Remark 4.2 & Theorem 4.1.

5. CONCLUSION

In the current article, the famous H-H like inequalities for $(m_1, m_2)-$conv. func. are obtained by implementing the widely recognized power-mean & Hölder's inequalities & implementations for theory of prob. & numerical integration are also deduced. We have captured various findings of several published articles [1,5] & also deduced few especial cases of $(m_1, m_2)-$conv. func.

References

[1]. S. S. Dragomir and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett., 11(5) (1998), 91—95.

[2]. Ali Hassan, Asif R. Khan, Faraz Mehmood and Maria Khan, Fuzzy Ostrowski Type Inequalities Via $\boldsymbol{\phi}-\boldsymbol{\lambda}-$Convex Functions, Journal of Mathematics and Computer Science, 28 (2023), 224-235.

[3]. Nazia Irshad, Asif R. Khan, Faraz Mehmood and Josip E. Pecaric, New Perspectives on the Theory of Inequalities for Integral and Sum, Birkhäuser (Springer Nature Switzerland), 2021.

[4]. Huriye Kadakal, $(\alpha, m_1, m_2)-$Convexity and some Inequalities of Hermite-Hadamard Type, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 68(2) (2019), 2128–2142.

[5]. Havva Kavurmaci, Merve Avci and M. E. Özdemir, new inequalities of hermite-hadamard type for convex functions with applications, Journal of Inequalities and Applications, 2011(86) (2011), 11 pages.

[6]. Asif R. Khan and Faraz Mehmood, Some Remarks on Functions with Non-decreasing Increments, Journal of Mathematical Analysis, 11(1) (2020), 1-16.

[7]. Faraz Mehmood, Asif R. Khan and Muhammad Adnan, Positivity of Integrals for Higher Order $\nabla-$Convex and Completely Monotonic Functions, Sahand Communications in Mathematical Analysis, 19(1) (2022), 119-137.

[8]. Faraz Mehmood, Faisal Nawaz, Sumaira Yousuf Khan and Akhmadjon Soleev, Hermite-Hadamard Like Inequalities for s−Convex Function in the 1st Kind & Applications, Journal of Liaoning Technical University (Natural Science Edition), 18(8) (2024), 134-142.

[9]. Faraz Mehmood, Muhammad Awais Shaikh, Syeda Ismat Zehra, Fouzia Abid and Mateen Ahmed Ghaznzvi, Inequalities of Hermite-Hadamard Like for s−Convex Function in the 2nd Kind and Implementations in Numerical Integration and Probability Theory, Dialogue Social Science Review (DSSR), 3(11) (2025), 37-48.